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JOURNAL OF
Economic
Dynamics
& Control

Journal of Economic Dynamics & Control 28 (2004) 1925–1954

www.elsevier.com/locate/econbase

Equilibrium stock return dynamics under alternative rules of learning about hidden states

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Received 6 August 2002; accepted 16 September 2003

Abstract

We examine the properties of equilibrium stock returns in an economy in which agents need to learn the hidden state of the endowment process. We consider Bayesian and suboptimal learning rules, including near-rational learning, conservatism, representativeness, optimism, and pessimism. Bayesian learning produces realistic variation in the conditional equity risk premium, return volatility, and Sharpe ratio. Alternative learning behaviors alter significantly the level and variation of the conditional return moments. However, when agents are allowed to be conscious of their learning mistakes and to price assets accordingly, the properties of returns under Bayesian and alternative learning rules are virtually indistinguishable.

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JEL classification: G0; G12; G14

Keywords: Time-varying moments or returns; Behavioral biases

1. Introduction

The equity risk premium is time-varying, and understanding why and how it varies is a lively research field. Intuitively, there are two reasons for the risk premium to vary in a rational expectations equilibrium (REE) framework: either the compensation required by agents to take on a marginal unit of risk (the market price of risk) changes or the amount of risk in the economy changes. It is relatively straightforward to generate endogenous changes in the market price of risk through changing aggregate

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preferences (induced, for example, by habit formation or heterogeneous agents), but it is more difficult to generate endogenous changes in the variance-covariance structure of a REE model. One mechanism is incomplete information, where agents must learn about unobservable features of the economy, such as parameters or latent state variables, from observables.¹ As agents become more or less sure about the true values of the unobservables, the uncertainty in the economy fluctuates and, as a result, the risk premium varies.

Relative to time-varying preferences, where the variation in the risk premium is essentially controlled by the researcher's modeling of the preferences, the incomplete information setting is significantly less flexible. The fact that there is only one way to learn optimally, namely through Bayesian updating, ties the researcher's hands. Rather than being a modeling choice, the learning process, which generates the time-variation of the risk premium in an economy with incomplete information, is fixed by the assumption of rational expectations.

Despite being optimal and therefore rational, Bayesian learning is not the only learning process advocated in the literature. In fact, it has recently become fashionable to explain empirical irregularities which are difficult to explain in a fully rational model through alternative forms of learning motivated by the psychology literature. For example, Barberis et al. (1998) and Brav and Heaton (2002) explain over- and under-reaction of stock prices to news with 'representativeness' and 'conservatism', where agents place too much or too little weight on the most recent data relative to Bayesian learning. Daniel et al. (1998, 2001) and Odean (1998) use 'overconfidence', where agents are too confident in the quality of private information, to explain the same phenomena. Cecchetti et al. (2000) resolve the equity risk premium puzzle with 'optimism' about the duration of recessions and 'pessimism' about the duration of expansions. Finally, Abel (2002) studies the effect of pessimism and 'doubt' on expected returns.

Since the learning process controls the dynamics of the risk premium in a REE model with incomplete information (but constant aggregate preferences) and there are a variety of alternative learning rules advocated in the finance literature, it is natural to consider the effects of these alternative learning rules on the dynamics of the risk premium. This is the aim of this paper. We conduct a systematic study of the quantitative effects of alternative learning, as opposed to Bayesian learning, on the conditional distribution of stock returns in an otherwise REE model. The three key features of our approach are:

- *Common economic model.* We study a variety of learning rules in the context of a common economic model. We consider a Lucas (1978) fruit-tree economy with identical agents that have recursive preferences (Epstein and Zin, 1989, 1991; Weil, 1989) and an exogenous endowment that follows a four-state Markov switching process. The agents know the structure of the economy and all of its parameters but cannot observe the current or past states of the economy. The only difference between the versions of this model we consider is the updating rule the agents

¹ Incomplete information models include Detemple (1986, 1991), Wang (1993a), Moore and Schaller (1996), David (1997), Brennan and Xia (1998), and Veronesi (1999, 2000).

use to incorporate new information into their beliefs about the hidden state of the economy.²

- *Broad set of learning rules.* The alternative learning rules we consider cover a broad spectrum of the literature on bounded rationality and learning: near-rational learning, in which the agents update their beliefs about the hidden state using Bayes' rule but occasionally make random mistakes; conservatism and representativeness, in which the agents update their beliefs with too little or too much emphasis on the most recent data; and optimism and pessimism, in which the agents systematically bias their beliefs toward or away from the good states.³
- *Distinction between ignorant and conscious agents.* We argue that there are two ways to introduce alternative learning into an otherwise REE model. The first is to assume that the agents follow a suboptimal learning rule but think that they learn optimally, which implies that the assets are priced the same way as in the Bayesian benchmark model except with different state-beliefs. We refer to these agents as ignorant, since they are unaware of their own limitations. The second way is to assume that the agents *knowingly* follow a suboptimal learning rule and account for this fact in setting the asset prices (effectively trying to compensate for or hedge against their learning mistakes). We refer to these agents as conscious and note that assuming irrational but conscious agents represents a far less severe breach of full rationality than irrational and ignorant agents. We further argue that consciousness can be justified from a costs versus benefits perspective of correcting either the learning behavior, which is an on-going effort, or the asset pricing rule, which involves only a one-time correction. It may well be optimal and rational for the agents to be consciously irrational.

Depending on whether one believes in the ideal of full rationality or not (we deliberately do not take a stance on this issue here), there are two ways to interpret the contributions of this paper. From a bounded rationality perspective, we compare a broad range of behavioral learning rules within a common economic model and study the implications of allowing agents to be conscious. Our results can be used to assess the equilibrium implications of a given behavioral learning rule or, from a reverse-engineering perspective, to determine which behavioral learning rule is best suited for matching the stylized features of the data. From a full-rationality perspective, we check the robustness of the incomplete information model to deviations from Bayesian learning. In that sense, our analysis contributes to the

² We impose the same learning rule uniformly on all agents, or equivalently on the representative agent, and hence disregard the interesting issue of how agents with different learning rules and/or heterogeneous beliefs interact and aggregate. The role of competing learning rules and heterogeneous beliefs is studied by Brock and Hommes (1997, 1998), Brock and LeBaron (1996), Detemple and Murthy (1994), LeBaron (2001), and Wang (1993b), among others. A related issue (which we also sidestep) is whether rational and irrational agents can co-exist in a competitive market. For this research topic, see Bernardo and Welch (2001), DeLong et al. (1990), Hirshleifer and Luo (2001), and Shleifer and Vishny (1997).

³ See Camerer (1995) and Conlisk (1996) for detailed surveys of this literature.

extensive literature on the robustness of REE models to deviations from optimal behavior.⁴

Our findings are easy to summarize. Bayesian learning performs reasonably well in matching the unconditional moments of stock returns and in producing realistic variation in the conditional equity premium, return volatility, and Sharpe ratio. Alternative learning of ignorant agents affects both the level and time-variation of the moments of stock returns. However, allowing agents to be conscious of their suboptimal learning behavior eliminates virtually all these differences in the return dynamics. This suggests that the benefits of considering alternative learning rules depend crucially on the assumption of ignorance.

The remainder of this paper is structured as follows. In Section 2, we set up the economic model and describe how asset prices are determined under full and incomplete information. Section 3 reviews Bayesian learning and formalizes the alternative learning rules. We present our quantitative results in Section 4 and conclude in Section 5.

2. Economic model

We consider a Lucas (1978) fruit tree economy populated by a large number of identical and infinitely lived individuals that can be aggregated into a single representative agent. The only source of income in the economy is a large number of identical and infinitely lived fruit trees. Without loss of generality, we assume that there exists one tree per individual, so that the amount of fruit produced by a tree in period t , denoted D_t , represents the output or dividend per capita. The fruits are non-storable and cannot be used to increase the number of trees. In equilibrium, all fruits are therefore consumed during the period in which they are produced, i.e., $C_t = D_t$, where C_t is the per-capita consumption in period t . Finally, we assume that each tree has a single perfectly divisible claim outstanding on it and that this claim can be freely traded at a price P_t in a competitive equity market.

The dividends are exogenously stochastic.⁵ We define $d_t \equiv \ln D_t$ and assume that the dividend growth rate $\Delta d_t \equiv d_t - d_{t-1}$ follows a Markov mean-switching process:⁶

$$\Delta d_t = \mu(S_{t-1}) + \sigma \varepsilon_t, \quad (1)$$

where ε_t is iid standard normal. S_t follows a finite-state Markov chain with transition matrix $\{p_{ij}\}_{N \times N}$, where N is the number of states and p_{ij} is the conditional probability

⁴ This literature includes Muth (1961), Akerlof and Yellen (1985a, b), Cochrane (1989), Day et al. (1974), Ingram (1990), Krusell and Smith (1996), Lettau and Uhlig (1999) and Wang (1993b).

⁵ Timmermann (1994) argues more generally that there may exist a feedback from stock prices to dividends which can lead to the existence of multiple rational expectations equilibria. Incorporating this feedback effect into our incomplete information framework is beyond the scope of this paper.

⁶ Cecchetti et al. (1990) provide empirical justification for modeling the dividend growth rate as a mean-switching process. Related models with similar endowment process include Abel (1994), Cecchetti et al. (1993), and Kandel and Stambaugh (1990, 1991). Driffill and Sola (1998) present evidence that the volatility of dividend growth is also state-dependent. However, to keep the model simple we keep the dividend growth volatility constant.

of the process being in state j next period given that it is in state i this period:

$$p_{ij} = \text{Prob}[S_{t+1} = j | S_t = i] \tag{2}$$

with $p_{ij} \in [0, 1]$. For notational convenience, we let $\mu(i)$ denote $\mu(S_t = i)$.

Following Epstein and Zin (1989) and Weil (1989), we assume that the preferences of the representative agent are defined recursively by

$$U_t = ((1 - \beta)C_t^{(1-\gamma)/\theta} + \beta(E_t[U_{t+1}^{1-\gamma}])^{1/\theta})^{\theta/(1-\gamma)}, \tag{3}$$

where $\beta \leq 1$ is the subjective discount factor, $\gamma > 0$ is the coefficient of relative risk aversion, and $\theta = (1 - \gamma)/(1 - (1/\psi))$ with $\psi > 0$ being the elasticity of intertemporal substitution. The first-order condition with Epstein–Zin–Weil preferences can be expressed as

$$E_t \left[\left(\beta \left(\frac{C_{t+1}}{C_t} \right)^{-1/\psi} R_{t+1} \right)^\theta \right] = 1, \tag{4}$$

where $R_{t+1} \equiv (P_{t+1} + D_{t+1})/P_t$ denotes the return on the market portfolio.

If the agents have full information (i.e., know the structure of the economy, its parameters, and the current state S_t) we can solve for the equilibrium asset price P_t by the method of undetermined coefficients (see Appendix A for details). Specifically, the price-dividend ratio λ_t and the risk-free rate R_t^f take on only N values, $\lambda(i)$ and $R^f(i)$, for $i = 1, 2, \dots, N$. In more realistic economies in which the process S_t is unobservable and the agents learn about the current state of the economy, the price-dividend ratio and risk-free rate are continuous functions. Intuitively, they are convex combinations of the full-information values.

Consider economies with incomplete information in which the agents know the structure and parameters of the model but do not observe the state variable S_t .⁷ Formally, the agents know that Δd_t follows the Markov switching process in Eq. (1) with parameters $\mu(i)$ and σ and with transition probabilities p_{ij} . However, the agents must form an opinion about the probability that the economy is currently in any particular state using the information filtration generated by the observed dividend series $\mathcal{F}_t = \{d_0, d_1, \dots, d_t\}$ and a set of updating rules for their subjective beliefs (such as Bayes' rule). The agents' subjective probability assessment $\pi_t \equiv \{\pi_t(1), \pi_t(2), \dots, \pi_t(N - 1)\}$, where $\pi_t(i) \equiv \text{Prob}[S_t = i | \mathcal{F}_t]$, determines the demands for the assets and, through market-clearing, sets their equilibrium prices.

To price the risky asset, we conjecture a solution of the form $P_t = \lambda_t D_t$, where the price-dividend ratio λ_t now depends on the subjective state-belief π_t as well as on the observed dividend growth rate Δd_t . From the first-order condition in Eq. (4), the price-dividend ratio satisfies the equation

$$\lambda(\pi_t, \Delta d_t)^\theta = \beta^\theta E_t \left[\left(\frac{D_{t+1}}{D_t} \right)^{\theta - (\theta/\psi)} (\lambda(\pi_{t+1}, \Delta d_{t+1}) + 1)^\theta \right]. \tag{5}$$

⁷ We do not consider learning about the parameters or structure of the model. The role of learning about parameters, considered by Detemple (1986, 1991), Timmermann (1993, 1996), and Cassano (1999), among others, is asymptotically degenerate, unless the true model changes periodically.

Since the agents use Δd_{t+1} to form the belief π_{t+1} , the two terms in the expectation are not independent and Eq. (5) generally does not have an analytical solution.⁸ We solve the model numerically using the projection method of Judd (1992) (see Appendix B for details).

3. Learning

3.1. Bayesian learning

The benchmark case of Bayesian learning works as follows. The agents leave period $t - 1$ with the information \mathcal{F}_{t-1} summarized by the subjective belief π_{t-1} . Once the dividend D_t is observed, the agents use Bayes' theorem to update their beliefs to π_t . The updating is simplified by the fact that the current state S_t has no contemporaneous effect on D_t . As a result, the agents use the newly observed data only to update their beliefs about the state for the *previous* period, denoted ${}_t\pi_{t-1}(i) \equiv \text{Prob}[S_{t-1} = i | \mathcal{F}_t]$, and then use the transition probabilities p_{ij} to form their beliefs $\pi_t(i) \equiv \text{Prob}[S_t = i | \mathcal{F}_t]$ about the *current* state.

Formally, starting at the end of period $t - 1$ with the subjective and so-called prior belief π_{t-1} , the agent enters period t and observes the new information D_t , or equivalently $\Delta d_t \equiv d_t - d_{t-1}$. From the mean-switching specification in Eq. (1), the probability density function of Δd_t conditional on the information at time $t - 1$ is

$$f(\Delta d_t | S_{t-1} = i, \mathcal{F}_{t-1}) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{(\Delta d_t - \mu(i))^2}{2\sigma^2} \right]. \tag{6}$$

We define

$$\alpha_t(i) \equiv f(\Delta d_t | S_{t-1} = i, \mathcal{F}_{t-1}) \pi_{t-1}(i) \tag{7}$$

and let ${}_t\pi_{t-1}^B(i)$ denote the updated belief $\text{Prob}[S_{t-1} = i | \mathcal{F}_t]$ under Bayesian learning. The updating is done optimally through Bayes' rule:

$${}_t\pi_{t-1}^B(i) = \frac{\text{Prob}[\Delta d_t | S_{t-1} = i, \mathcal{F}_{t-1}] \times \text{Prob}[S_{t-1} = i | \mathcal{F}_{t-1}]}{\text{Prob}[\Delta d_t | \mathcal{F}_{t-1}]} = \frac{\alpha_t(i)}{\sum_{j=1}^N \alpha_t(j)}. \tag{8}$$

Finally, the agents combine the output from the updating step in Eq. (8) with the transition probabilities p_{ij} to form the Bayesian belief $\pi_t^B(i) \equiv \text{Prob}[S_t = i | \mathcal{F}_t]$ about the current state:

$$\pi_t^B(i) = \sum_{j=1}^N p_{ji} \times {}_t\pi_{t-1}^B(j). \tag{9}$$

⁸ In the nested case of Bayesian learning with power utility, the price-dividend ratio is available analytically. Veronesi (2000) provides the solution in a continuous time model and David and Veronesi (2001) solve the corresponding discrete time model. Specifically, the price-dividend ratio λ_t is a belief-weighted average of the $\lambda(i)$ that solve the first-order condition under full information.

3.2. Alternative learning rules

We now turn to alternative learning behavior, which is suboptimal and in some cases even biased relative to the optimal Bayesian learning. Consistent with our representative agent framework, we impose the same suboptimal learning rule uniformly on all agents or equivalently on the representative agent (see footnote 2).

3.2.1. Near-rational learning

The first suboptimal learning rule we consider is near-rational learning, in which the agents update their beliefs about the hidden state using Bayes' rule but occasionally make mistakes. The mistakes are assumed to be random in such a way that the subjective belief π_t is still conditionally unbiased, meaning that the agents do not deviate from the benchmark case of Bayesian learning *on average*. Formally, we maintain that

$$E[\pi_t | \mathcal{F}_t] = \pi_t^B. \tag{10}$$

where π_t^B is the Bayesian belief about the state. This unbiasedness property distinguishes near-rational learning from the other alternative learning rules which are biased.

We formalize near-rational learning as follows. Once the agents observe the dividend D_t they update their prior belief π_{t-1} about the previous state to ${}_t\pi_{t-1}$ not through Bayes' rule but instead through a weighted average of Bayes' rule and a random error term:

$${}_t\pi_{t-1}(i) = (1 - \omega) {}_t\pi_{t-1}^B(i) + \omega \eta_t(i), \tag{11}$$

where $\eta_t(i)$ denotes a random error with state-dependent distribution, the weight ω , which is assumed to be state-independent, takes a value in $[0, 1]$, and ${}_t\pi_{t-1}^B(i)$ denotes the Bayesian updating *process* (as opposed to the Bayesian belief) described in Eq. (8).⁹ Given the updated belief about the state in the previous period, the belief about the current state is again formed using the transition probabilities:

$$\pi_t(i) = \sum_{j=1}^N p_{ji} \times {}_t\pi_{t-1}(j). \tag{12}$$

We need to impose more structure on the random noise term η_t to guarantee that the posterior beliefs ${}_t\pi_{t-1}$ are valid probabilities and sum to one across states. Specifically, we assume that for a *fixed* benchmark state i the error $\eta_t(i)$ follows a Beta distribution with parameters $\alpha_t(i)$ and $\sum_{j \neq i} \alpha_t(j)$, where the vector α_t is defined in Eq. (7). Moreover, we assume that the errors are perfectly correlated across all states and that for state j other than the benchmark state i :

$$\eta_t(j) = (1 - \eta_t(i)) \frac{\alpha_t(j)}{\sum_{j \neq i} \alpha_t(j)}. \tag{13}$$

⁹ We distinguish between the Bayesian updating process and Bayesian belief, which is the outcome of the Bayesian updating process when used in conjunction with the Bayesian belief from the previous period. By acting on the updating process, an error feeds into all futures periods because the contaminated belief serves as prior for the next period. If the error acts directly on the belief, its effects lasts only one period.

This particular way of distributing the error across the states guarantees that the resulting beliefs π_t satisfy the unbiasedness condition in Eq. (10).¹⁰ It is also straightforward to verify that ${}_t\pi_{t-1} \in [0, 1]$ for all states and that it sums to one across states.

3.2.2. Conservatism and representativeness

Conservatism and representativeness are psychologically motivated alternatives to Bayesian learning (Edwards, 1968; Kahneman and Tversky, 1972) that have recently attracted attention in behavioral finance.¹¹ Conservatism leads individuals to place too much emphasis on old data or the status-quo and too little emphasis on recent data or the possibility of change.¹² Representativeness refers to the exact opposite behavior, that individuals tend to think relatively short data sequences are representative of the underlying distribution.

To formalize conservatism and representativeness, we assume that the agents update their beliefs π_{t-1} about the state in the previous period to ${}_t\pi_{t-1}$ not through Bayes rule but instead through the following updating rule:

$${}_t\pi_{t-1}(i) = (1 - \omega) {}_t\pi_{t-1}^B(i) + \omega \pi_{t-1}(i) \quad (14)$$

for conservatism and

$${}_t\pi_{t-1}(i) = (1 - \omega) {}_t\pi_{t-1}^B(i) + \omega \frac{f(\Delta d_t | s_{t-1} = i)}{\sum_{j=1}^N f(\Delta d_t | s_{t-1} = j)} \quad (15)$$

for representativeness, where ω is a parameter that takes a value in $[0, 1]$.

For conservatism, the updated belief in Eq. (14) is a convex combination of the Bayesian belief and the prior belief. Since Bayesian updating reflects an *optimal* weighting of the likelihood of the data Δd_t and the prior belief π_{t-1} [Eqs. (7) and (8)], conservative agents place more weight on their prior belief and less weight on the data in the updating process. The parameter ω measures the degree of conservatism. Analogously, the updated belief in Eq. (15) for representativeness is a convex combination of the Bayesian belief and the likelihood of the data. Agents that suffer from representativeness place less weight on the prior belief and more weight on the data than Bayesian agents. Note that conservatism and representativeness lead to conditionally biased state-beliefs.

¹⁰ Recall that if y follows a Beta distribution with parameters $\{\beta, \alpha\}$, $E[y] = \beta/(\beta + \alpha)$ and $\text{Var}[y] = \beta\alpha/((\beta + \alpha + 1)(\beta + \alpha)^2)$. It then follows that $E[\eta_t(i) | \mathcal{F}_t] = \alpha_t(i) / \sum_{i=1}^N \alpha_t(i)$.

¹¹ See, for example, DeBondt and Thaler (1985), Lakonishok et al. (1994), Barberis et al. (1998) and Brav and Heaton (2002).

¹² Conservatism can also be interpreted as ‘overconfidence’ in one’s ability to learn. Overconfidence typically refers to individuals placing too much weight on their private information relative to the public information. Although there is no real distinction between public and private information in our model, the dividend realizations are obviously public while the prior beliefs are arguably private. Conservative agents place too much weight on their prior belief because they irrationally overrate its accuracy. It therefore appears that conservative agents are overconfident in their ability to learn.

3.2.3. Optimism and pessimism

Optimistic agents systematically bias their beliefs toward good states and pessimistic agents tend to think the economy is in bad states.¹³ We define good states to be states with $\mu(i) \geq \bar{\mu}$, where $\bar{\mu}$ denotes the unconditional *median* dividend growth rate, and bad states to be states with $\mu(i) < \bar{\mu}$. We order the state index i such that states with larger indices correspond to higher conditional mean dividend growth rates, which means that good states correspond to the indices $i \geq N/2$ and bad states to $i < N/2$.¹⁴

To capture the notion of optimism, we remove mass of the Bayesian posterior beliefs from the bad states, in proportion to the conditional probabilities of being in each of the bad states, and then distribute this mass, again proportionally, across the good states. Formally, we define the optimistic beliefs as

$$\pi_t(i) = \begin{cases} (1 - \omega)\pi_t^B(i) + \omega \frac{\pi_t^B(i)}{\sum_{j \geq N/2} \pi_t^B(j)} & \text{for } i \geq \frac{N}{2} \text{ (good states),} \\ (1 - \omega)\pi_t^B(i) & \text{for } i < \frac{N}{2} \text{ (bad states),} \end{cases} \quad (16)$$

where ω , which takes a value in $[0,1]$, measures the degree of optimism. For pessimism, we remove mass from the good states and distribute it proportionally across the bad states:

$$\pi_t(i) = \begin{cases} (1 - \omega)\pi_t^B(i) & \text{for } i \geq \frac{N}{2} \text{ (good states),} \\ (1 - \omega)\pi_t^B(i) + \omega \frac{\pi_t^B(i)}{\sum_{j < N/2} \pi_t^B(j)} & \text{for } i < \frac{N}{2} \text{ (bad states),} \end{cases} \quad (17)$$

where, in this case, ω measures the degree of pessimism.

3.3. Ignorant versus conscious learning

There are two ways to introduce alternative learning into an otherwise REE model. The first is to assume that the agents follow a suboptimal learning rule but think that they learn optimally. We refer to these agents as ignorant because they are unaware of their limitations. With irrational and ignorant agents the assets are priced by the same price–dividend ratio $\lambda(\pi_t)$ as with rational agents. Therefore, the only differences between the two versions of the model are the realizations and the evolution of the state-beliefs. Conditional on the same belief realization $\pi_t = \pi_t^B$, they are identical. The second way to introduce alternative is to assume that the agents *knowingly* follow a suboptimal learning rule and incorporate this fact into asset prices by using a different

¹³ On one hand, psychological studies find that people tend to be optimistic about their future prospects (Weinstein, 1980; Kunda, 1987) and that, perhaps somewhat counter-intuitively, optimism is more pronounced among more intelligent people (Klaczynski and Fauth, 1996). On the other hand, Cecchetti et al. (2000) and Abel (2002) show that pessimistic behavior can help explain the equity premium and risk-free rate puzzles, respectively.

¹⁴ The classification of states according to the median growth rate is notationally convenient. Alternatively, we could use the mean growth rate, but the two approaches are equivalent for our quantitative results.

price–dividend ratio function to compensate for or hedge against their learning mistakes. We refer to these agents as conscious.

To intuitively understand how irrational but conscious agents can partially correct their learning mistakes through the price–dividend ratio function, consider an economy with two states $S_t = \{0, 1\}$ and a pessimistic agent with state-beliefs that are unconditionally biased $E[\pi_t] = 0.9E[\pi_t^B]$. Furthermore, assume that the price–dividend ratio for the Bayesian agents is $\lambda^B(\pi_t) = 100 + 50\pi_t^B$. In this example, irrational and ignorant agents underprice the asset (relative to the dividends) by an average of $E[\lambda_t] - E[\lambda_t^B] = -5E[\pi_t^B]$. Irrational but conscious agents, in contrast, recognize that with their particular learning mistakes a price–dividend ratio of $\lambda(\pi_t) = 100 + 55.66\pi_t$ results in unconditionally unbiased valuations. However, unless the belief of the irrational agents is proportional to that of the rational agents in all states, which is not the case with the alternative learning rules described above, conscious agents cannot fully correct their mistakes through the price–dividend ratio. Intuitively, they can only compensate for systematic biases through their own price–dividend ratio function.¹⁵

Technically, conscious agents solve for the function $\lambda(\pi_t, \Delta d_{t+1})$ that satisfies the first-order condition in Eq. (5) when the conditional expectations are taken with respect to the dynamics of their suboptimal state-beliefs. Ignorant agents, in contrast, use the price–dividend ratio function of the Bayesian agents, which solves the first-order condition when the conditional expectations are taken with respect to the dynamics of the Bayesian beliefs.

Perhaps the most intuitive reason for considering conscious agents is internal consistency. Since the agents in our model are infinitely lived, it is difficult to imagine that they employ a set of learning and pricing rules that consistently misprices the asset relative to the historical dividend realizations. At the same time, psychologists argue convincingly that it is difficult for naturally pessimistic individuals, for example, not to be pessimistic in their probability assessment. One can interpret conscious agents as adjusting their pricing rule to be at least partially consistent with the data.

One way to justify why conscious agents adjust their pricing rule to be internally consistent rather than correct their learning behavior, is from a costs versus benefits perspective. As Simon (1955), Marschak (1968), and Einhorn (1970, 1971) suggest, computational costs are an important consideration in deciding whether to act rationally or according to a behavioral heuristic (see also Payne et al., 1990). It is arguably less costly for agents to adjust their pricing rule *once* than to correct and monitor *each period* (for an infinite number of periods) their natural tendency to be pessimistic. As long as the expected utility loss from being irrational but conscious does not exceed the costs of not being pessimistic, it can therefore be optimal (and rational) to be irrational but conscious.

¹⁵ This intuition is not quite correct due to the non-linearities in the first-order condition (5). Unless $\theta = 1$ (power utility), conscious agents may also be able to partially correct for too much or too little variation in their state-belief. The price–dividend ratio can also be different due to a correlation between dividend growth realizations and the learning errors, which generates positive or negative hedging demands for the asset (e.g., Merton, 1969). However, the first-order corrections are for systematic biases in the belief.

4. Quantitative results

4.1. Calibration

We estimate the parameters of the mean-switching process in Eq. (1) by maximum likelihood using quarterly real dividends paid on the Standard and Poors Composite Index from January 1871 to December 1998 (512 observations). We use a four-state regime-switching model for the dividend growth rate. The parameter estimates are reported in Table 1.

All four states are quite persistent, with continuation probabilities (the diagonal p_{ii}) ranging from 0.64 to 0.84. The extreme states (states one and four) are less persistent than the moderate states (states two and three). The first two states are contraction states with negative average dividend growth rates and the last two states are expansion states with positive mean dividend growth rates. The conditional volatility of dividend growth is 1.9% per quarter, which together with the variation in the conditional mean growth rate, translates into an unconditional volatility of 6.8% per year.

To get a sense for how difficult it is to learn about the latent state, we plot in Fig. 1 the conditional dividend growth densities scaled by the unconditional state probabilities. We also plot the implied unconditional density, which is the sum of the four scaled conditional densities. Intuitively, the more the conditional densities overlap, the harder it is to determine the state of the economy from a dividend growth realization. For example, if a dividend growth is realized from the first state, it is immediately clear that the economy is not in the third or fourth state. However, there could be some uncertainty about whether the economy is in states one or two. It is therefore relatively easy to learn about state one. In contrast, the conditional densities of states two and three overlap substantially, which means that these two states are very difficult to distinguish and learn about.

Table 1
Maximum likelihood estimates of the endowment process

	$i = 1$	$i = 2$	$i = 3$	$i = 4$	μ_i	σ
<i>Conditional probabilities and moments</i>						
$j = 1$	0.636	0.046	0.006	0.000	-0.073	0.019
$j = 2$	0.327	0.844	0.153	0.065	-0.005	
$j = 3$	0.038	0.087	0.841	0.164	0.020	
$j = 4$	0.000	0.023	0.001	0.771	0.069	
<i>Unconditional probabilities and moments</i>						
	0.072	0.519	0.356	0.053	0.003	0.034

This table presents maximum likelihood estimates of the log dividend growth rate process:

$$\Delta d_t = \mu(S_{t-1} = i) + \sigma \varepsilon_t,$$

where ε_t is iid $N[0,1]$ and S_t follows a four-state Markov switching process with transition probabilities $\{p_{ij}\}_{4 \times 4}$ of $S_t = j$ given $S_{t-1} = i$. The data are quarterly dividend growth rates for the Standard and Poors Composite Index from January 1871 to December 1999 (512 observations).

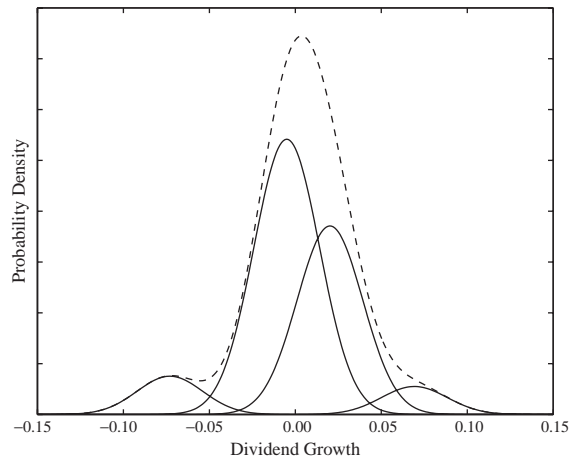


Fig. 1. Probability densities of dividend growth. This figure plots the four conditional probability densities weighted by the corresponding unconditional state probabilities (solid lines) and the unconditional probability density (dashed line) of the Markov mean-switching dividend growth rate process.

We need to specify the preference parameters to price assets with this endowment process. Rather than present results for a variety of different parameter values, we focus on a single choice which produces reasonable return dynamics under Bayesian learning. However, our findings, especially the results on alternative learning and on the role of ignorance versus consciousness, are qualitatively the same for various other sensible parameter configurations. (These results are available on request.) We choose a relative risk aversion of $\gamma = 4$, an elasticity of intertemporal substitution of $\psi = 2$, and a time-preference coefficient of $\beta = 0.99$. The choice of γ and β is fairly standard and, as we will demonstrate below, generates realistic levels of the equity premium and risk-free rate under Bayesian learning. We choose an elasticity of intertemporal substitution that exceeds one, which is unusual (recall that with power utility $\psi = 1/\gamma$), to match the stylized fact that stock prices are high relative to dividends in expansions and low in recessions.¹⁶

Given the estimates in Table 1 and these preference parameters, we can solve for the full-information price-dividend ratios, which correspond to the extreme beliefs $\pi_t =$

¹⁶ The reciprocal of the price-dividend ratio, the dividend yield, is counter-cyclical. See, for example, Fama and French (1989). Bansal and Yaron (2003) use $\psi = \{2.5, 4\}$ for the same reason. As Cecchetti et al. (1990) explain, the cyclicity of the price-dividend ratio depends on the relative importance of two offsetting effects: an intertemporal relative price effect and a substitution effect. The intertemporal relative price effect is that the agents try to buy the stock to save for future consumption at high endowments (since the states are persistent) and low relative prices. The intertemporal substitution effect is that agents try to sell the stock to increase their current consumption in anticipation of future high endowments. When the elasticity of intertemporal substitution is less than one, the intertemporal substitution effect dominates, and the price-dividend ratio is counter-cyclical. When the elasticity of intertemporal substitution is greater than one, the relative price effect dominates, and the price-dividend ratio is cyclical (as in the data). When $\gamma = \psi = 1$ (log utility) the two effects exactly cancel and the price-dividend ratio is constant.

$\{[1\ 0\ 0\ 0], [0\ 1\ 0\ 0], [0\ 0\ 1\ 0], [0\ 0\ 0\ 1]\}$. The results are $\lambda_t = \{40.65, 45.15, 47.54, 52.12\}$. As expected (from our discussion above), the price-dividend ratios increase monotonically with the mean dividend growth rate. Furthermore, the ratios for states two and three are much closer to each other [$\lambda(3) - \lambda(2) = 2.39$] than for states one and two [$\lambda(2) - \lambda(1) = 4.50$] and for states three and four [$\lambda(4) - \lambda(3) = 4.58$], which means that the stock price fluctuates more when the economy moves between a moderate and an extreme state than when it moves between the two moderate states. This pattern in the price-dividend ratios turns out to be quite important for understanding the results in Section 4.5.

Finally, we need to calibrate the tuning parameter ω in the alternative learning rules. For near-rational learning, we choose ω in Eq. (11) to fix the conditional *standard deviation* of the mode of the state-distribution ($\pi_t(2)$ in our case) around the corresponding Bayesian belief to be 0.025, 0.050, or 0.075 (recall that the errors are perfectly correlated across states).¹⁷ In the case of conservatism and representativeness, the tuning parameter ω in Eqs. (14) and (15) can be interpreted as the relative deviation of the learning rule from Bayesian learning. We report results for $\omega = 0.05, 0.10, \text{ and } 0.15$. For optimism and pessimism, we calibrate ω in Eqs. (16) and (17) such that the unconditional bias of the belief that the economy is in a good state $E[\pi_t(3) + \pi_t(4)] - E[\pi_t^B(3) + \pi_t^B(4)]$ equals 0.01, 0.025, and 0.05 for optimism and $-0.01, -0.025, \text{ and } -0.05$ for pessimism.

4.2. Research methodology

Given the calibrated model and the preference parameters, we numerically solve for the price-dividend ratio function $\lambda(\pi_t)$. We then evaluate the unconditional and conditional moments of returns as follows:

1. We simulate 1000 time-series of 532 quarters of the latent state variable, where we initialize each time-series with the unconditional state probabilities. We then simulate a dividend growth realization from every state.
2. For each learning rule, we construct 1000 sequences of state beliefs by applying the corresponding updating steps to the dividend growth rate series from step 1.
3. To evaluate the unconditional moments of returns for a given learning rule, we generate 1000×1024 returns using the factorization:

$$R_{t+1} = \frac{\lambda(\pi_{t+1}) + 1}{\lambda(\pi_t)} \frac{D_{t+1}}{D_t}, \tag{18}$$

where the price-dividend ratios are evaluated at the state beliefs from step 2 and the dividend growth rates come from step 1. We drop the first 512 quarters in each series as a burn-in (resulting in 512 quarters, which matches our sample) and then evaluate the unconditional moments with their sample analogs for these 1000×512 returns.

4. To evaluate the conditional moments for a given learning rule and a given state belief π_t from step 2, we simulate 5000 *one-period ahead* returns using steps analogous

¹⁷ Fixing the standard deviation of the belief for the other states produces qualitatively similar results.

to steps 1–3, except that the states are initialized with the state probabilities π_t . We then evaluate the conditional moments with their sample analogs for these 5000 returns.

For ignorant agents we compute the returns by evaluating in step 3 the price-dividend ratio function of the Bayesian agents at the distorted beliefs, as ignorant agents think they act optimally and hence use the pricing rule of optimally acting (i.e., Bayesian) agents. For conscious agents, in contrast, we solve for a separate price-dividend ratio function for each learning rule. These alternative pricing rules satisfy the first-order conditions in Eq. (5) when the expectations are evaluated with the dynamics of the alternative state beliefs. In that sense, the agents are conscious of the fact that their learning is not optimal.

4.3. State-beliefs

Table 2 reports the across simulation averages and standard deviations of the conditional beliefs for the four states π_t and for bad states $\pi_t(1) + \pi_t(2)$.¹⁸ The table also describes the price-dividend ratios $\lambda(\pi_t)$ (assuming ignorant agents). The first row shows the statistics for Bayesian learning. The average beliefs match closely the unconditional probabilities of the four states from Table 1. All four beliefs exhibit considerable variability, ranging from 14.5% for state one to 25.3% for state three, which means that agents form conditional beliefs that are at times very different from their unconditional counterparts. This is because the realized dividend growth rates are quite informative about the state.

Rows 2–4 describe the beliefs for near rational learning. The averages are close to those for Bayesian learning, except for a slight shift of probability mass from state three to state two (the mode of the distribution). Intuitively, the learning errors offset the information contained in the data and pull the beliefs toward the unconditionally most likely state.¹⁹ Increasing the standard deviation of the learning errors raises the variability of the beliefs, with the beliefs for the moderate states becoming proportionally more variable. The price-dividend ratios are unaffected on average but become, as expected, more variable as the standard deviation of the learning errors increases.

Conservatism (in rows 5–7) leads to beliefs that are approximately unbiased and become less variable as the degree of conservatism increases. By overweighing their prior beliefs the agents learn more slowly (hence lower variability) but ultimately come to the same conclusions as with Bayesian learning (hence unbiasedness). The variability of the price-dividend ratios also decreases as the degree of conservatism increases.

Although representativeness (in rows 8–10) is theoretically the opposite of conservatism, its effects on the beliefs are not symmetric. Representativeness shifts probability mass from the mode of the state distribution, the most likely second state, to the other states, particularly the extreme ones. Just like conservatism, it thereby reduces the

¹⁸ The average belief for good states $\pi_t(3) + \pi_t(4)$ is one minus the average for bad states shown in the table. The standard deviations of the beliefs for good and bad states are therefore the same.

¹⁹ This pattern is independent of whether we calibrate the near-rational learning rule to state two or three.

Table 2
Conditional beliefs and price-dividend ratios

	$\pi_t(1)$		$\pi_t(2)$		$\pi_t(3)$		$\pi_t(4)$		$\pi_t(1) + \pi_t(2)$		$\lambda(\pi_t)$	
	Avg	Std	Avg	Std	Avg	Std	Avg	Std	Avg	Std	Avg	Std
Bayesian	0.073	0.150	0.520	0.247	0.354	0.252	0.053	0.153	0.593	0.279	45.86	1.55
<i>Near rational</i>												
$\sigma[\pi] = 2.5\%$	0.073	0.150	0.521	0.253	0.353	0.259	0.053	0.153	0.594	0.284	45.86	1.56
$\sigma[\pi] = 5.0\%$	0.073	0.152	0.522	0.270	0.350	0.277	0.054	0.154	0.595	0.301	45.86	1.58
$\sigma[\pi] = 7.5\%$	0.073	0.153	0.526	0.295	0.345	0.302	0.054	0.155	0.598	0.324	45.82	1.59
<i>Conservatism</i>												
$\omega = 5\%$	0.073	0.146	0.520	0.239	0.354	0.244	0.053	0.150	0.592	0.271	45.85	1.51
$\omega = 10\%$	0.073	0.141	0.519	0.231	0.355	0.236	0.054	0.146	0.592	0.264	45.85	1.48
$\omega = 15\%$	0.073	0.136	0.519	0.222	0.355	0.228	0.054	0.142	0.592	0.256	45.84	1.44
<i>Representativeness</i>												
$\omega = 5\%$	0.073	0.151	0.515	0.242	0.356	0.245	0.056	0.153	0.588	0.275	45.87	1.55
$\omega = 10\%$	0.074	0.151	0.510	0.237	0.358	0.239	0.058	0.154	0.584	0.273	45.89	1.56
$\omega = 15\%$	0.074	0.151	0.506	0.233	0.360	0.234	0.060	0.155	0.580	0.270	45.90	1.57
<i>Optimism</i>												
Bias = 1.0%	0.071	0.148	0.502	0.240	0.374	0.250	0.053	0.153	0.573	0.273	45.91	1.53
Bias = 2.5%	0.068	0.144	0.476	0.231	0.402	0.246	0.053	0.152	0.545	0.265	45.98	1.51
Bias = 5.0%	0.060	0.132	0.402	0.201	0.485	0.232	0.052	0.151	0.462	0.238	46.20	1.44
<i>Pessimism</i>												
Bias = 1.0%	0.074	0.150	0.538	0.240	0.335	0.241	0.053	0.150	0.612	0.269	45.80	1.51
Bias = 2.5%	0.075	0.150	0.564	0.230	0.310	0.225	0.051	0.144	0.639	0.255	45.73	1.46
Bias = 5.0%	0.079	0.150	0.637	0.202	0.236	0.176	0.046	0.126	0.717	0.212	45.50	1.31

This table describes the conditional state-beliefs π_t and the price-dividend ratios $\lambda(\pi_t)$ for different learning rules. We simulate 1000 samples of 512 quarterly observations of the state-beliefs. We then report the across simulations averages and standard deviations of the conditional beliefs and price-dividend ratios.

variability of the moderate state-beliefs, but, unlike conservatism, it raises the variability of the extreme state-beliefs. The reason for this asymmetry is that, while the agents can slow down the process of Bayesian learning (through conservatism), they cannot speed it up because Bayesian learning already incorporates new information as fast and efficiently as possible. As a result, the beliefs with representativeness tend to ‘overshoot’ the moderate states and settle on the less likely extreme states. This leads to price-dividend ratios that increase in both average magnitude and variability as the degree of representativeness increases.

Optimism (in rows 11–13) and pessimism (in rows 14–16) affect the beliefs symmetrically. Optimism shifts probability mass from the bad states to the good ones, particularly to the third state, and pessimism shifts mass from the good states to the bad ones, particularly to the second state. As the degree of optimism (pessimism) increases, optimism (pessimism) thereby generates higher (lower) price-dividend ratios on

Table 3
Unconditional moments

	Ignorant learning				Conscious learning			
	r_{t+1}		r_t^f		r_{t+1}		r_t^f	
	Avg	Std	Avg	Std	Avg	Std	Avg	Std
Data	11.04	18.18	4.88	2.80	11.04	18.18	4.88	2.80
Bayesian	9.79	10.02	2.18	2.77	9.79	10.02	2.18	2.77
	<i>Near rational</i>				<i>Near rational</i>			
$\sigma[\pi] = 2.5\%$	9.79	10.05	2.26	2.80	9.79	9.79	1.98	2.97
$\sigma[\pi] = 7.5\%$	9.79	10.24	2.74	3.04	9.79	9.85	1.57	2.56
	<i>Conservatism</i>				<i>Conservatism</i>			
$\omega = 5\%$	9.79	9.88	1.89	2.68	9.79	9.95	2.07	3.02
$\omega = 15\%$	9.79	9.59	1.35	2.49	9.79	10.16	2.13	2.96
	<i>Representativeness</i>				<i>Representativeness</i>			
$\omega = 5\%$	9.79	10.12	2.15	2.78	9.79	10.05	2.09	3.00
$\omega = 15\%$	9.78	10.32	2.11	2.81	9.79	10.16	2.13	2.96
	<i>Optimism</i>				<i>Optimism</i>			
Bias = 1.0%	9.78	10.00	2.39	2.82	9.79	10.01	2.07	3.10
Bias = 5.0%	9.72	9.83	4.08	2.73	9.79	10.09	2.15	2.91
	<i>Pessimism</i>				<i>Pessimism</i>			
Bias = 1.0%	9.80	9.97	1.82	2.60	9.79	10.02	2.06	3.03
Bias = 5.0%	9.85	9.60	0.29	1.90	9.79	10.12	2.05	3.03

This table describes the unconditional moments of stock and bond returns in the data and implied by the model for different learning rules and with ignorant and conscious agents (except for Bayesian learning). We simulate 1000 samples of 512 quarterly stock and bond returns and report the overall averages and standard deviations of the continuously compounded stock returns $r_{t+1} = \ln R_{t+1}$ and bond returns $r_t^f = \ln R_t^f$.

average. Both learning behaviors reduce the variability of the beliefs and the price-dividend ratios because it requires a more informative dividend growth realization to overwrite the optimistic or pessimistic bias.

4.4. Unconditional moments

We now turn to the unconditional moments of the resulting equilibrium stock returns. The left-hand panel of Table 3 reports the unconditional moments of stock and bond returns in the historical data and implied by the model for Bayesian learning and alternative learning with ignorant agents. The second row shows that with Bayesian learning our model is able to generate reasonable levels of the equity premium and riskfree rate, 7.6% and 2.2% per year, respectively.²⁰ The volatility of interest rates is

²⁰ However, we note that none of the models solves the equity premium puzzle. The reason is that we use dividend growth, as opposed to consumption growth, to be the exogenous driving process.

a low 2.8% per year, but the volatility of stock returns is too low at only 10% per year, as opposed to 18% in the historical data. This means that with the parameters estimated from the dividend data, the model cannot fully overcome the ‘excess volatility’ puzzle documented by Grossman and Shiller (1981), among others.

Since near rational learning introduces idiosyncratic noise to the beliefs, it increases the volatility of stock returns. As the stock returns become more volatile, so does the pricing kernel [see Eq. (A.4)] and, as a result, the average risk-free rate increases from 2.2% for Bayesian learning to 2.7% with $\sigma(\pi_t) = 0.075$. The average stock return is unchanged, so that the equity risk premium drops from 7.6% to 7.1%.

The effects of conservatism and representativeness on the volatility of stock returns are intuitive. With increasing conservatism, the state-beliefs become less responsive to the dividends and hence less volatile, causing the stock returns to also become less volatile. With increasing representativeness, in contrast, the beliefs become more (excessively) sensitive to the dividends, which results in a higher volatility of stock returns.

Optimistic agents systematically over-estimate the probability of being in good states. For them not to short-sell the bond to buy more stock (recall that with $\psi > 1$ the agents want to buy stock in good states to save for future consumption at high endowments and low relative prices), the riskfree rate increases with the degree of optimism. This causes the equity risk premium to drop. The effect of pessimism is exactly the opposite. Pessimistic agents under-estimate the probability of being in bad states and want to shift wealth from stocks to bonds. The riskfree rate therefore decreases and the equity risk premium increases.²¹ The volatility of stock returns decreases slightly with both optimism and pessimism, reflecting the fact that the beliefs become less volatile.

We now compare the proceeding results for ignorant agents with those for conscious agents in order to gauge the importance of implicitly assuming ignorance. The right panel of Table 3 reports the unconditional moments under conscious learning. We focus on conservatism, representativeness, optimism, and pessimism, as these rules have received much attention in the literature. The results are striking. Under conscious learning, the average stock return remains the same as that in Bayesian benchmark case, and the differences in the average riskfree rate are much less pronounced as with ignorant learning.

4.5. Conditional moments of returns

4.5.1. Equity risk premium, conditional volatility, and Sharpe ratio

Turning to the core of our inquiry, the effects of alternative learning on the conditional moments of returns, we first present the results for Bayesian learning and then discuss the differences induced by alternative learning relative to this benchmark. The right panel of Table 4 describes the conditional moments of stock returns in the data and implied by the model. The historical estimates are obtained using the approach of Brandt and Kang (2003). For the model, we report the across time and simulations

²¹ This result is consistent with Abel (2002) and Cecchetti et al. (2000) who show that moderate degrees of pessimism go a long way towards resolving the equity risk premium and riskfree rate puzzles.

Table 4
Conditional moments

	Ignorant learning								Conscious learning							
	$E_t[r_{t+1}^e]$		$\sigma_t[r_{t+1}^e]$		$E_t[r_{t+1}^e]/\sigma_t[r_{t+1}^e]$				$E_t[r_{t+1}^e]$		$\sigma_t[r_{t+1}^e]$		$E_t[r_{t+1}^e]/\sigma_t[r_{t+1}^e]$			
	Avg	Std	Avg	Std	Avg	Std	ρ_1	ρ_4	Avg	Std	Avg	Std	Avg	Std	ρ_1	ρ_4
Data	7.27	1.93	12.55	3.06	0.64	0.18	0.70	0.23	7.27	1.93	12.55	3.06	0.64	0.18	0.70	0.23
Bayesian	7.60	2.70	9.67	1.97	0.77	0.21	0.63	0.25	7.60	2.70	9.67	1.97	0.77	0.21	0.63	0.25
<i>Near rational</i>								<i>Near rational</i>								
$\sigma[\pi] = 2.5\%$	7.52	2.96	9.71	1.95	0.76	0.25	0.65	0.25	7.81	3.15	9.58	2.10	0.81	0.29	0.66	0.25
$\sigma[\pi] = 7.5\%$	7.05	4.98	9.93	1.85	0.68	0.51	0.55	0.17	8.22	3.09	9.46	2.18	0.87	0.32	0.35	0.10
<i>Conservatism</i>								<i>Conservatism</i>								
$\omega = 5\%$	7.90	2.60	9.51	1.92	0.82	0.20	0.57	0.21	7.72	3.14	9.55	2.09	0.80	0.29	0.68	0.27
$\omega = 15\%$	8.44	2.78	9.18	1.80	0.92	0.23	0.33	0.08	7.66	3.31	9.69	2.16	0.78	0.27	0.57	0.19
<i>Representativeness</i>								<i>Representativeness</i>								
$\omega = 5\%$	7.63	2.91	9.80	1.93	0.76	0.23	0.55	0.21	7.70	3.22	9.68	2.05	0.78	0.29	0.66	0.25
$\omega = 15\%$	7.67	3.44	10.02	1.87	0.75	0.29	0.38	0.13	7.66	3.31	9.82	2.01	0.77	0.29	0.59	0.21
<i>Optimism</i>								<i>Optimism</i>								
Bias = 1.0%	7.39	2.82	9.65	1.99	0.75	0.21	0.63	0.25	7.72	3.22	9.62	2.09	0.79	0.29	0.67	0.26
Bias = 5.0%	5.64	2.87	9.43	2.07	0.57	0.21	0.52	0.19	7.64	3.31	9.69	2.16	0.78	0.27	0.57	0.19
<i>Pessimism</i>								<i>Pessimism</i>								
Bias = 1.0%	7.98	2.47	9.62	1.96	0.82	0.18	0.57	0.20	7.73	3.27	9.63	2.07	0.79	0.30	0.68	0.26
Bias = 5.0%	9.56	2.21	9.22	1.90	1.04	0.17	0.41	0.10	7.74	3.57	9.76	2.05	0.78	0.36	0.62	0.20

This table describes the conditional moments of stock and bond returns in the data and implied by the model for different learning rules and with ignorant and conscious agents (except for Bayesian learning). We simulate for each of the 1000×512 state-beliefs 5000 one-period ahead stock and bond returns and compute the conditional risk premium and volatility of stock returns. We then report the across imulations averages and standard deviations of the conditional risk premium, return volatility, and Sharpe ratio. For the Sharpe ratio, we also report the across simulations averages and standard deviations of the quarterly and annual autocorrelations.

averages and standard deviations of the conditional risk premium, conditional volatility, and conditional Sharpe ratio. We also report the across simulations average first and fourth autocorrelations of the Sharpe ratio.

Consider first the results for Bayesian learning in the second row. The standard deviation of the conditional equity risk premium is 2.70%, which means that the risk premium varies considerably through time. The average conditional volatility of returns is 9.67%, somewhat lower than the unconditional volatility due to the variation of the risk premium, with a standard deviation of 1.97%.²² Together, the conditional moments imply a Sharpe ratio of 0.77 on average with a standard deviation of 0.21. Despite its variability, the Sharpe ratio is quite persistent, with a quarterly autocorrelation of 0.63

²² Recall that $\text{Var}[x] = E[\text{Var}_t[x]] + \text{Var}[E_t[x]]$.

and an annual autocorrelation of 0.25. Finally, notice that all of these quantities match reasonably well their empirical counterparts.

Near rational learning has some effect on the variation of the conditional risk premium and Sharpe ratio. The standard deviation of the risk premium rises, the standard deviation of the Sharpe ratio more than doubles, and the persistence of the Sharpe ratio drops significantly.

With conservatism, the beliefs adjust more sluggishly to new dividend realizations. This causes the conditional risk premium to increase (because the riskless rate decreases as the pricing kernel becomes less volatile) and the conditional return volatility to decrease on average. The average Sharpe ratio therefore rises sharply. Furthermore, conservatism not only causes the conditional moments to become less variable, since the beliefs vary less, but also weakens the link between the conditional moments. This can be seen as the persistence of the Sharpe ratio drops from a first-order autocorrelation of 0.57 to 0.33, suggesting that the risk premium and volatility vary more independently.

As we explained in Section 4.3, representativeness causes the beliefs to overshoot the moderate states and settle on the less likely extreme states. This has the expected effect of increasing the conditional and unconditional volatility of returns. It also has the somewhat puzzling effect, when compared to the results for near rational learning, of increasing the conditional and unconditional risk premium. The reason is that with representativeness, the beliefs visit more often extreme regions in the state-space, in which the risk premium is significantly higher as compensation for an increased return volatility. The average risk premium therefore increases, while the average Sharpe ratio is almost unchanged. Since part of the risk premium in the extreme states is compensation for a high or low consumption beta, the conditional risk premiums fluctuates more than the conditional return volatility. As a result, the conditional Sharpe ratio becomes a little more volatile.

The results for optimism and pessimism are much more straightforward. With optimism (pessimism) the average conditional risk premium decreases (increases) because of the counter-cyclical variation of the covariance of returns with consumption growth. Optimistic agents inflate the probability of being in low consumption beta states (states with a relatively low risk premium) and the opposite for pessimistic agents. In both cases, the average conditional return volatility decreases because the state-beliefs become less variable (it takes a more informative dividend realization to overwrite the learning bias). As a result, the average Sharpe ratio drops with optimism and rises with pessimism.

To compare these results for ignorant agents with those for conscious agents, the right panel of Table 4 describes the conditional return moments for conscious agents. The results are again striking. In all cases, the effects of alternative learning on the level of the equity risk premium documented in the left panel of the same table disappear when we allow the agents to be conscious. Consider, for example, conservatism. With ignorant agents the average risk premium rises from 7.60% to 8.44%, but with conscious agents it rises only marginally to 7.66%. Similarly for pessimism, with conscious agents the average risk premium rises to only 7.74% instead of 9.56%. Note that consciousness causes the risk premium to become somewhat more variable. This is most apparent for pessimism, where with ignorant agents the standard deviation of

the conditional risk premium drops from 2.70% to 2.21%, but with conscious agents it rises to 3.57%.

The effects of consciousness on the conditional volatility of returns are similar but less extreme. For example, in the case of conservatism the average conditional volatility drops from 9.67% to 9.18% assuming ignorance but rises slightly to 9.69% allowing for consciousness. In the case of pessimism, the average conditional volatility drops to 9.22% with ignorance but actually increases slightly to 9.76% with consciousness. As with the conditional risk premium, consciousness causes the volatility to be somewhat more variable.

Combining the results for the conditional risk premium and volatility, the level of the Sharpe ratio is hardly affected by alternative learning with conscious agents. However, since both conditional moments become more variable, the Sharpe ratio also becomes somewhat less persistent and more variables.

The conclusion from these results is obvious. Allowing agents to be conscious of their learning limitations and to price assets optimally with respect to their suboptimal beliefs changes dramatically the role of alternative learning. A more general implication of our results is that behavioral models which use suboptimal learning rules to explain certain return anomalies may well be relying more on the implicit assumption of ignorance than on the particular form of the suboptimal learning rule.

4.5.2. *Predictive regressions*

It is common in the literature to capture variation in the conditional equity risk premium through predictive regressions. Perhaps the most popular regressor is the dividend-price ratio (e.g., Campbell and Shiller, 1988; Fama and French, 1988). To examine the extent to which our model captures return predictability by the dividend yield and, more importantly, the extent to which alternative learning affects this predictability, we compute predictive regressions for each simulated return series. Table 5 presents the across simulations averages and standard deviations of the slope coefficients for regressions of (continuously compounded) one-, three-, five-, and ten-year excess returns on the log dividend yield. The table also reports the average R^2 of the predictive regressions.

Bayesian learning generates a realistic level of predictability. As in the data, the average R^2 increases initially with the return horizon, to almost 4% for 3-year returns, and then decreases at longer horizons. The slope coefficients increase monotonically with the horizon. Judging by the t -statistics (not reported to preserve space), the slope coefficients are statistically significant on average, but the average coefficients are in all cases less than 1.5 sampling standard deviations from zero.

Comparing the results across the rows in Table 5 illustrate clearly again that (i) suboptimal learning with ignorant agents alters the return properties from the Bayesian benchmark; and then (ii) these effects are largely eliminated by consciousness. Without going into details of every suboptimal learning rule, we focus our discussion here on pessimism, which is intriguing in light of the recent interest in this alternative learning rule for explaining the equity premium and riskless rate puzzles (e.g., Abel, 2002; Cecchetti et al., 2000). The results in Tables 3 and 4 verify that pessimism raises the unconditional and average conditional risk premium and, at the same time, lowers the

Table 5
Long-horizon predictive regressions

	1 Year			3 Years			5 Years			10 Years		
	Avg	Std	\bar{R}^2	Avg	Std	\bar{R}^2	Avg	Std	\bar{R}^2	Avg	Std	\bar{R}^2
<i>Ignorant learning</i>												
Data	0.10	na	0.023	0.32	na	0.061	0.64	na	0.091	1.16	na	0.135
Bayesian	0.41	0.29	0.030	0.66	0.47	0.037	0.72	0.57	0.034	0.82	0.77	0.032
<i>Near rational</i>												
$\sigma[\pi] = 2.5\%$	0.43	0.29	0.032	0.68	0.48	0.038	0.74	0.57	0.035	0.84	0.77	0.032
$\sigma[\pi] = 7.5\%$	0.57	0.29	0.049	0.80	0.47	0.047	0.85	0.57	0.040	0.94	0.78	0.035
<i>Conservatism</i>												
$\omega = 5\%$	0.40	0.30	0.027	0.66	0.50	0.036	0.72	0.60	0.033	0.83	0.80	0.031
$\omega = 15\%$	0.33	0.32	0.021	0.62	0.54	0.030	0.69	0.66	0.029	0.81	0.89	0.029
<i>Representativeness</i>												
$\omega = 5\%$	0.41	0.28	0.030	0.63	0.48	0.035	0.69	0.57	0.033	0.79	0.76	0.031
$\omega = 15\%$	0.41	0.28	0.030	0.59	0.46	0.033	0.64	0.55	0.030	0.74	0.75	0.028
<i>Optimism</i>												
Bias = 1.0%	0.43	0.29	0.031	0.67	0.47	0.038	0.73	0.57	0.035	0.83	0.76	0.032
Bias = 5.0%	0.34	0.30	0.022	0.49	0.51	0.025	0.54	0.62	0.023	0.66	0.84	0.023
<i>Pessimism</i>												
Bias = 1.0%	0.33	0.29	0.022	0.53	0.50	0.027	0.59	0.61	0.026	0.70	0.82	0.026
Bias = 5.0%	-0.07	0.34	0.011	-0.09	0.64	0.012	-0.04	0.81	0.012	0.12	1.13	0.012
<i>Conscious learning</i>												
<i>Near rational</i>												
$\sigma[\pi] = 2.5\%$	0.50	0.28	0.039	0.78	0.46	0.047	0.84	0.55	0.042	0.93	0.73	0.038
$\sigma[\pi] = 7.5\%$	0.37	0.28	0.023	0.50	0.48	0.022	0.54	0.60	0.020	0.64	0.87	0.020
<i>Conservatism</i>												
$\omega = 5\%$	0.53	0.28	0.044	0.86	0.46	0.056	0.92	0.54	0.050	1.01	0.71	0.044
$\omega = 15\%$	0.56	0.28	0.046	0.93	0.45	0.062	1.00	0.54	0.056	1.09	0.71	0.049
<i>Representativeness</i>												
$\omega = 5\%$	0.50	0.28	0.040	0.78	0.45	0.048	0.84	0.54	0.044	0.93	0.72	0.039
$\omega = 15\%$	0.47	0.28	0.036	0.68	0.45	0.040	0.74	0.54	0.036	0.83	0.72	0.033
<i>Optimism</i>												
Bias = 1.0%	0.51	0.28	0.041	0.80	0.46	0.050	0.86	0.54	0.045	0.95	0.72	0.040
Bias = 5.0%	0.47	0.27	0.036	0.68	0.44	0.039	0.73	0.54	0.035	0.83	0.72	0.032
<i>Pessimism</i>												
Bias = 1.0%	0.51	0.29	0.042	0.81	0.46	0.051	0.87	0.55	0.046	0.96	0.72	0.041
Bias = 5.0%	0.48	0.29	0.038	0.71	0.46	0.042	0.76	0.55	0.038	0.86	0.73	0.0340

This table describes the predictive regressions of continuously compounded excess stock returns on a constant and the dividend-price ratio in the data and implied by the model for different learning rules with ignorant and conscious learning. The table shows the across simulations averages and standard deviations of the slope coefficients and the across simulations average R^2 s. The return horizon ranges from one to 10 years.

Table 6
Conditional volatility dynamics

	Autocorrelations				GARCH(1,1)				$\rho[\Delta\sigma_{t+1}, R_{t+1}^c]$	
	ρ_1		ρ_4		a		b			
	Avg	Std	Avg	Std	Avg	Std	Avg	Std	Avg	Std
<i>Ignorant learning</i>										
Data	0.758	na	0.332	na	0.136	na	0.810	na	-0.423	na
Bayesian	0.623	0.050	0.208	0.070	0.164	0.073	0.423	0.252	-0.372	0.097
<i>Near rational</i>										
$\sigma[\pi] = 2.5\%$	0.622	0.049	0.210	0.069	0.161	0.073	0.424	0.259	-0.371	0.096
$\sigma[\pi] = 7.5\%$	0.582	0.057	0.177	0.071	0.157	0.075	0.415	0.275	-0.333	0.098
<i>Conservatism</i>										
$\omega = 5\%$	0.652	0.045	0.225	0.070	0.171	0.073	0.406	0.252	-0.380	0.095
$\omega = 15\%$	0.675	0.043	0.234	0.071	0.183	0.074	0.379	0.253	-0.388	0.093
<i>Representativeness</i>										
$\omega = 5\%$	0.625	0.049	0.208	0.069	0.159	0.071	0.424	0.258	-0.358	0.096
$\omega = 15\%$	0.605	0.051	0.193	0.069	0.151	0.066	0.427	0.263	-0.344	0.097
<i>Optimism</i>										
Bias = 1.0%	0.638	0.047	0.219	0.069	0.166	0.074	0.423	0.254	-0.388	0.0949
Bias = 5.0%	0.641	0.046	0.216	0.069	0.182	0.077	0.416	0.237	-0.442	0.0909
<i>Pessimism</i>										
Bias = 1.0%	0.630	0.048	0.215	0.070	0.164	0.073	0.420	0.256	-0.375	0.096
Bias = 5.0%	0.614	0.054	0.195	0.071	0.173	0.072	0.409	0.243	-0.405	0.093
<i>Conscious learning</i>										
<i>Near rational</i>										
$\sigma[\pi] = 2.5\%$	0.621	0.051	0.208	0.071	0.191	0.086	0.407	0.238	-0.351	0.102
$\sigma[\pi] = 7.5\%$	0.603	0.055	0.188	0.072	0.212	0.100	0.384	0.252	-0.327	0.109
<i>Conservatism</i>										
$\omega = 5\%$	0.653	0.047	0.221	0.072	0.195	0.084	0.396	0.239	-0.354	0.101
$\omega = 15\%$	0.678	0.045	0.231	0.073	0.209	0.085	0.376	0.238	-0.359	0.104
<i>Representativeness</i>										
$\omega = 5\%$	0.615	0.051	0.204	0.070	0.184	0.082	0.409	0.246	-0.339	0.100
$\omega = 15\%$	0.593	0.054	0.190	0.040	0.178	0.078	0.420	0.241	-0.321	0.101
<i>Optimism</i>										
Bias = 1.0%	0.631	0.049	0.213	0.071	0.189	0.083	0.408	0.238	-0.360	0.100
Bias = 5.0%	0.635	0.048	0.212	0.071	0.196	0.084	0.411	0.232	-0.400	0.099
<i>Pessimism</i>										
Bias = 1.0%	0.627	0.051	0.210	0.071	0.186	0.083	0.405	0.253	-0.347	0.101
Bias = 5.0%	0.612	0.055	0.193	0.073	0.181	0.079	0.412	0.239	-0.322	0.106

This table describes the dynamics of the conditional volatility of excess stock returns in the data and implied by the model for different learning rules with ignorant and conscious learning. We report the across simulations averages and standard deviations of the quarterly and annual autocorrelations of the conditional volatility. The table also shows the across simulations averages and standard deviations of the coefficient of a GARCH(1,1) for the conditional volatility. Finally, we report the across simulations averages and standard deviations of the correlation between changes in the conditional volatility and excess stock returns.

riskless rate. But, the results in Table 5 suggest that pessimism with ignorant learning also eliminates most long-horizon predictability by the dividend yield. It therefore appears as if pessimism replaces the equity premium and riskfree rate puzzles with a dividend yield predictability puzzle. Finally, this counterfactual pattern is completely obliterated once the agent is allowed to be conscious, as is, unfortunately, the success of pessimism in Tables 3 and 4.

4.5.3. Conditional heteroskedasticity

Table 6 takes a closer look at the time-series properties of the conditional volatility. We present quarterly and annual autocorrelations of the conditional volatility in the data and implied by the model to measure the degree of volatility clustering, where the results for the data are again based on the conditional volatility estimates obtained using the approach of Brandt and Kang (2003).²³ As a more sophisticated way to measure volatility clustering, we also fit the following GARCH(1,1) model to data and the simulated return series:

$$\sigma_t^2 = c + a\varepsilon_{t-1}^2 + b\sigma_{t-1}^2, \quad (19)$$

where σ_t is the conditional volatility and ε_t denotes the excess return innovation. For the model, we report the across simulations average and standard deviation of the estimates of a and b . Finally, we consider the contemporaneous correlation between changes in the conditional volatility and excess returns, denoted $\rho[r_{t+1}^e, \Delta\sigma_{t+1}]$. In the data, this correlation is negative and large in magnitude, especially at the daily and weekly frequency, and is typically attributed to the ‘leverage’ effect.

The conditional volatility dynamics induced by Bayesian learning are again quite realistic. The conditional volatility is highly persistent, with a quarterly autocorrelation of 0.623. The annual autocorrelation of 0.208 is significantly higher than the value of 0.147 implied by geometric decay of the quarterly autocorrelation, which is consistent with the recent evidence of long-memory in volatility.²⁴ The GARCH(1,1) coefficients are consistently positive across simulations with average values of $a = 0.164$ and $b = 0.423$. Finally, the correlation between changes in volatility and returns is consistently negative with an average value of -0.37 .

The intuition underlying the volatility clustering is simple. The return volatility depends on the degree of uncertainty about the state of the economy. Since this uncertainty is resolved gradually through learning, periods of high volatility (great uncertainty) and period of low volatility (little uncertainty) are clustered through time. The negative correlation between changes in volatility and returns is equally intuitive. Recall that expected returns are high in times of high volatility and low in times of low volatility. Therefore, when the return volatility increases the stock price must drop for the expected return to increase, and the opposite when the volatility decreases.

Given this intuition, understanding the effects of the alternative learning rules on the conditional volatility dynamics is straightforward. In particular, near rational learning

²³ Volatility clustering refers to the fact that returns exhibit prolonged periods of persistently high and persistently low volatility, so that large returns (positive or negative) tend to be followed by large returns.

²⁴ See, for example, Ding et al. (1993), and Baillie et al. (1996).

and representativeness cause the beliefs to be less persistent and hence reduce the degree of volatility clustering. Conservatism, optimism, and pessimism lead to more sluggish beliefs, which enhances volatility clustering. Similarly for the leverage effect. Near rational learning and conservatism shift subjective probability mass toward the center of the state-distribution, creating less uncertainty about states one and four, where the positive relationship between the risk premium and volatility (or the negative relationship between returns and changes in volatility) is most pronounced, and thereby dampen the leverage effect. Optimism and pessimism result in more extreme beliefs and, in particular, more frequent transitions between state three and uncertainty about state four (with optimism) or between state two and uncertainty about state one (with pessimism). These learning rules therefore enhance the leverage effect. Notice that the effect of optimism is much stronger than that of pessimism. The reason is that, as we discussed above, with optimism both the conditional risk premium and volatility become more variable, while with pessimism they become less variable.

Overall, we conclude from these results that the volatility dynamics are relatively *insensitive* to the learning rule, with either ignorant or conscious agents. The quarterly autocorrelation of the conditional volatility with Bayesian learning is 0.62; it only ranges from 0.67 with conservatism to 0.58 with near rational learning and representativeness for ignorant agents. The range is even narrower for conscious agents. Similarly for the correlation between changes in volatility and return, which is -0.37 in the Bayesian benchmark and only ranges from -0.34 with representativeness to -0.44 with optimism for ignorant agents.

5. Conclusion

We conduct a systematic study of the properties of equilibrium stock returns in an incomplete information economy, in which agents need to learn the hidden state of the endowment process. We consider optimal Bayesian learning as well as a broad set of suboptimal alternative learning rules, including near rational learning, conservatism, representativeness, optimism, and pessimism. We also differentiate between alternative learning with ignorant agents, who are unaware that they learn suboptimally and hence price assets as if they were Bayesian, and with conscious agents, who realize that they learn suboptimally and account for this fact in setting asset prices.

Our findings are easy to summarize. Bayesian learning performs reasonably well in matching the unconditional moments of stock returns and in producing realistic variation in the conditional equity premium, return volatility, and Sharpe ratio. Alternative learning of ignorant agents affects both the level and time-variation of the moments of stock returns. However, allowing agents to be conscious of their suboptimal learning behavior eliminates virtually all these differences in the return dynamics. This suggests that the benefits of considering alternative learning rules depend crucially on the assumption of ignorance.

Our findings beg the question of whether agents are ignorant or conscious in reality. Although this question is ultimately for psychologists to answer, we suspect that the truth lies somewhere between the two extremes. If that is the case, understanding the

role of conscious agents is just as important as understanding the role of ignorant agents, which has until now been the primary focus of the literature.

Acknowledgements

We thank Andy Abel, Alon Brav, John Campbell, Wouter Den Haan, Simon Gervais, Joao Gomes, Cars Hommes (the editor), Leonid Kogan, Bart Moore, Jacob Sagi, Pietro Veronesi, Amir Yaron, two anonymous referees, as well as seminar participants at the 2001 meeting of the Society of Computational Economics, 2001 meeting of the Western Finance Association, 2001 fall meeting of the NBER asset pricing group, and the University of Pennsylvania for their helpful comments and suggestions. Financial support from the Rodney L. White Center for Financial Studies at the Wharton School is gratefully acknowledged.

Appendix A. Solution to the full-information economy

We conjecture the following solution:

$$P_t = \lambda(S_t)D_t. \tag{A.1}$$

Substituting Eq. (A.1) and the consumption policy $C_t = D_t$ into Eq. (4), we have²⁵

$$\lambda(S_t)^\theta = \beta^\theta E_t \left[\left(\frac{D_{t+1}}{D_t} \right)^{\theta - (\theta/\psi)} (\lambda(S_{t+1}) + 1)^\theta \right]. \tag{A.2}$$

Next, we write Eq. (1) in levels: $D_{t+1} = D_t \exp[\mu(S_t) + \sigma \varepsilon_{t+1}]$ and substitute it into Eq. (A.2) to obtain a set of N equations in the N unknown price-dividend ratios $\lambda(i)$, for $i = 1, 2, \dots, N$:

$$\lambda(i)^\theta = \beta^\theta \sum_{j=1}^N p_{ij} (1 + \lambda(j))^\theta \exp \left[\left(\theta - \frac{\theta}{\psi} \right) \mu(i) + \frac{1}{2} \sigma^2 \left(\theta - \frac{\theta}{\psi} \right)^2 \right]. \tag{A.3}$$

In general, these equations are non-linear and need to be solved numerically. In the special case of power utility, the equations are linear and the system has an analytical solution.²⁶ In either case, the fact that there exists an analytical or numerical solution verifies our initial conjecture that Eq. (A.1) solves the first-order condition (4).

If there exists a risk-free asset, its return is given by $1/E_t[M_{t+1}]$, where the intertemporal marginal rate of substitution (or pricing kernel) is

$$M_{t+1} = \left(\beta \left(\frac{C_{t+1}}{C_t} \right)^{-1/\psi} \right)^\theta (R_{t+1})^{\theta-1}. \tag{A.4}$$

²⁵ We use the fact that $R_{t+1} = (\lambda_{t+1}D_{t+1} + D_{t+1})/(\lambda_t D_t) = ((\lambda_{t+1} + 1)/\lambda_t)(D_{t+1}/D_t)$.

²⁶ See, for example, Cecchetti et al. (1990).

Substituting Eq. (A.1) and the consumption policy $C_t = D_t$ into Eq. (A.4) and taking conditional (on S_t) expectations yields, for $i = 1, 2, \dots, N$,

$$\frac{1}{R^f(i)} = \frac{\beta^\theta}{\lambda(i)^{\theta-1}} \sum_{j=1}^N p_{ij} (1 + \lambda(j))^{\theta-1} \times \exp \left[\left(\theta - 1 - \frac{\theta}{\psi} \right) \mu(i) + \frac{1}{2} \sigma^2 \left(\theta - 1 - \frac{\theta}{\psi} \right)^2 \right], \tag{A.5}$$

where $\lambda(i)$ is given by the solution to the set of Eqs. (A.3).

Appendix B. Application of the projection method

This section describes the application of the projection method of Judd (1992) to our model. For illustrative purposes, we only present the case of CRRA utility and two states of the world. Extension to the case with Epstein–Zin preference and multiple states is straightforward.

Start with the equilibrium price-dividend ratio given by the Euler equation:

$$\beta E_t [e^{\Delta d_{t+1}(1+\gamma)} [\lambda(\pi_{t+1}, \Delta d_{t+1}) + 1]] - \lambda(\pi_t, \Delta d_t) = 0, \tag{B.1}$$

where Δd_t is the exogenous state variable defined by

$$\Delta d_t = \alpha_0 + \alpha_1 S_{t-1} + \sigma \varepsilon_t \tag{B.2}$$

and where π_t is the endogenous state variable.

Under the benchmark case of Bayesian learning, the law of motion for π_t^B is given by

$$\pi_{t+1}^B = p(\pi_{t+1}^B) + (1 - q)(1 - \pi_t^B), \tag{B.3}$$

where

$$\pi_{t+1}^B = \frac{\exp \left[-\frac{(\Delta d_{t+1} - \alpha_0 - \alpha_1)^2}{2\sigma^2} \right] \pi_t}{\exp \left[-\frac{(\Delta d_{t+1} - \alpha_0)^2}{2\sigma^2} \right] (1 - \pi_t) + \exp \left[-\frac{(\Delta d_{t+1} - \alpha_0 - \alpha_1)^2}{2\sigma^2} \right] \pi_t}. \tag{B.4}$$

We assume that the pricing functional has the form $\lambda(\pi_t, \Delta d_t) = \lambda(\pi_t)$. In words, the price-dividend ratio does not depend on the exogenous state variable Δd_t directly.²⁷

We seek a function $\hat{\lambda}(\cdot)$ which depends on a finite-dimensional vector of parameters and which approximates the conditional expectation in the Euler Eq. (B.1). For this, we construct

$$\hat{\lambda}(\pi; \phi) \equiv \sum_{i=0}^n \phi_i \psi_i(\pi), \tag{B.5}$$

²⁷ We have actually carried out the two-dimensional numerical algorithm for λ as a bivariate function of both π_t and Δd_t . The numerical solution verifies our assumption that λ does not depend on Δd_t explicitly.

where $\{\psi_i(\pi)\}_{i=0}^n$ is a basis of complete Chebyshev polynomials of order n and ϕ_i are the coefficients of the polynomials. We let ϕ denote the vector of these coefficients.²⁸

Next, we define the residual equation as the Euler Eq. (B.1) evaluated at the approximate solution $\hat{\lambda}(\pi_t; \phi)$:

$$R(\pi_t; \phi) = \beta E_t[\exp(\Delta d_{t+1})]^{1+\gamma} [\hat{\lambda}(\pi_{t+1}; \phi) + 1] - \hat{\lambda}(\pi_t; \phi). \tag{B.6}$$

Plugging $\hat{\lambda}(\pi_t; \phi)$ from Eq. (B.1) and Δd_t from Eq. (B.2) into Eq. (B.6), we obtain

$$\hat{R}(\pi_t; \phi) = - \sum_{i=0}^n \phi_i \psi_i(\pi_t) + \beta E_t \times \left[\exp[(1 + \gamma)(\alpha_0 + \alpha_1 S_t + \sigma \varepsilon_{t+1})] \left[\sum_{i=1}^n \phi_i \psi_i(\pi_{t+1}^B) + 1 \right] \right], \tag{B.7}$$

where the expectation E_t is taken with respect to ε_{t+1} and S_t . Denote the integrand of the expectation in (B.6) as $f(S_t, \varepsilon_{t+1})$ and then rewrite the expectation as

$$E_t[f(S_t, \varepsilon_{t+1})] = \int [\pi_t f(1, \varepsilon_{t+1}) + (1 - \pi_t) f(0, \varepsilon_{t+1})] d\Phi(\varepsilon),$$

where $\Phi(\varepsilon)$ is the cdf of the standard normal distribution. In addition

$$f(1, \varepsilon_{t+1}) \equiv \exp[(1 + \gamma)(\alpha_0 + \alpha_1 + \sigma \varepsilon_{t+1})] \left[\sum_{i=1}^n \phi_i \psi_i(\pi_{t+1}^B(1)) + 1 \right], \tag{B.8}$$

$$f(0, \varepsilon_{t+1}) \equiv \exp[(1 + \gamma)(\alpha_0 + \sigma \varepsilon_{t+1})] \left[\sum_{i=1}^n \phi_i \psi_i(\pi_{t+1}^B(0)) + 1 \right] \tag{B.9}$$

and by Eq. (B.3)

$$\pi_{t+1}^B(1) \equiv p_{[t+1]} \pi_t^B(1) + (1 - q)[1 - {}_{t+1} \pi_t^B(1)], \tag{B.10}$$

$$\pi_{t+1}^B(0) \equiv p_{[t+1]} \pi_t^B(0) + (1 - q)[1 - {}_{t+1} \pi_t^B(0)]. \tag{B.11}$$

Now the residual function becomes

$$\hat{R}(\pi_t; \phi) = - \sum_{i=0}^n \phi_i \psi_i(\pi_t) + \beta \int [\pi_t f(1, \varepsilon_{t+1}) + (1 - \pi_t) f(0, \varepsilon_{t+1})] d\Phi(\varepsilon) \tag{B.12}$$

We want to choose ϕ such that $R(\pi_t; \phi)$ is as close to zero as possible for all values of π_t . The projection method sets the residual close to zero in the weighted integral sense. We use the orthogonal collocation. We choose ϕ such that the residual is set to be exactly zero at n points called collocation points. Since our basis functions are chosen from a set of orthogonal polynomials, the collocation points are given as the roots of the n th order Chebyshev polynomials. $\hat{R}(\pi_t; \phi)$ evaluated at these collocation points then gives a system of non-linear equations, which is in turn solved by a minimization routine. As a final note, we evaluate the integral contained in the residual, $\int [\pi_t f(1, \varepsilon_{t+1}) + (1 - \pi_t) f(0, \varepsilon_{t+1})] d\Phi(\varepsilon)$, numerically using Gauss–Hermite quadrature.

²⁸ In our implementation we use $n = 10$. The state space for the endogenous state variable π_t is obviously $[0, 1]$. The univariate Chebyshev polynomials are given by, $\psi_i^c(x) = \cos[i \arccos(2x - 1)]$ for $i = 1, 2, \dots, n$.

For the economies with ignorant agents who use alternative rules of learning, the equilibrium pricing functional is the same as in the benchmark Bayesian case because the agents think they follow Bayesian learning. Conscious agents, in contrast, understand the effects of their non-Bayesian learning and therefore take these effects into account in making their optimal consumption and saving decisions, which in turn changes the equilibrium asset prices. Simple modifications of the Bayesian solution algorithm suffices for conscious learning. In particular, for near rational learning, conservatism, and representativeness, we only need to replace ${}_{t+1}\pi_t^B(i)$, $i = 0, 1$ in the right-hand sides of Eqs. (B.10) and (B.11) with alternative learning counterparts. For optimism and pessimism, we only need to replace $\pi_{t+1}^B(i)$, $i = 0, 1$ in the right hand sides of Eqs. (B.8) and (B.9) with their alternative learning counterparts.

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